NATIONAL BUREAU OF STANDARDS REPORT

1996

CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES

by

P. Erdös and G. A. Hunt

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

U. S. DEPARTMENT OF COMMERCE
Charles Sawyer, *Secretary*

NATIONAL BUREAU OF STANDARDS
A. V. Astin, *Director*

THE NATIONAL BUREAU OF STANDARDS

The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work. In general, each section is engaged in specialized research, development, and engineering in the field indicated by its title. A brief description of the activities, and of the resultant reports and publications, appears on the inside of the back cover of this report.

Electricity. Resistance Measurements. Inductance and Capacitance. Electrical Instruments. Magnetic Measurements. Applied Electricity. Electrochemistry.

Optics and Metrology. Photometry and Colorimetry. Optical Instruments. Photographic Technology. Length. Gage.

Heat and Power. Temperature Measurements. Thermodynamics. Cryogenics. Engines and Lubrication. Engine Fuels. Cryogenic Engineering.

Atomic and Radiation Physics. Spectroscopy. Radiometry. Mass Spectrometry. Solid State Physics. Electron Physics. Atomic Physics. Neutron Measurements. Infrared Spectroscopy. Nuclear Physics. Radioactivity. X-Rays. Betatron. Nucleonic Instrumentation. Radiological Equipment. Atomic Energy Commission Instruments Branch.

Chemistry. Organic Coatings. Surface Chemistry. Organic Chemistry. Analytical Chemistry. Inorganic Chemistry. Electrodeposition. Gas Chemistry. Physical Chemistry. Thermochemistry. Spectrochemistry. Pure Substances.

Mechanics. Sound. Mechanical Instruments. Aerodynamics. Engineering Mechanics. Hydraulics. Mass. Capacity, Density, and Fluid Meters.

Organic and Fibrous Materials. Rubber. Textiles. Paper. Leather. Testing and Specifications. Polymer Structure. Organic Plastics. Dental Research.

Metallurgy. Thermal Metallurgy. Chemical Metallurgy. Mechanical Metallurgy. Corrosion.

Mineral Products. Porcelain and Pottery. Glass. Refractories. Enameled Metals. Concreting Materials. Constitution and Microstructure. Chemistry of Mineral Products.

Building Technology. Structural Engineering. Fire Protection. Heating and Air Conditioning. Floor, Roof, and Wall Coverings. Codes and Specifications.

Applied Mathematics. Numerical Analysis. Computation. Statistical Engineering. Machine Development.

Electronics. Engineering Electronics. Electron Tubes. Electronic Computers. Electronic Instrumentation.

Radio Propagation. Upper Atmosphere Research. Ionospheric Research. Regular Propagation Services. Frequency Utilization Research. Tropospheric Propagation Research. High Frequency Standards. Microwave Standards.

Ordnance Development.
Electromechanical Ordnance.
Electronic Ordnance.
These three divisions are engaged in a broad program of research and development in advanced ordnance. Activities include basic and applied research, engineering, pilot production, field testing, and evaluation of a wide variety of ordnance matériel. Special skills and facilities of other NBS divisions also contribute to this program. The activity is sponsored by the Department of Defense.

Missile Development. Missile research and development: engineering, dynamics, intelligence, instrumentation, evaluation. Combustion in jet engines. These activities are sponsored by the Department of Defense.

- Office of Basic Instrumentation
- Office of Weights and Measures.

NATIONAL BUREAU OF STANDARDS REPORT

NBS PROJECT **NBS REPORT**

1101-10-5100 August 29, 1952 1996

CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES *

by

P. Erdős

National Bureau of Standards, Los Angeles
and
University College of London

and

G. A. Hunt

National Bureau of Standards, Los Angeles
and
Cornell University

PREPRINT

* The preparation of this paper was sponsored (in part) by the Office of Naval Research, USN.

The publication, reprin part, is prohibited unless permission is obt ıdards, Washington 25, D. C. Such permis rt has been specifically prepared if that a rt for its own use.

Approved for public release by the Director of the National Institute of Standards and Technology (NIST) on October 9, 2015

CHANGES OF SIGN OF SUMS OF RANDOM VARIABLES *

by

P. Erdos** and G. A. Hunt ***

1. Let $x_1, x_2, \cdots$ be independent random variables all having the same continuous symmetric distribution, let $s_k = x_1 + \cdots + x_k$, and let N_n be the number of changes of sign in the sequence $s_1, \cdots, s_{n+1}$.

Th 1. $$\sum_{k=1}^{n} \frac{1}{2(k+1)} \leq E\{N_n\} \leq \frac{1}{2} \sum_{k=1}^{n} \phi(k)$$

Here $E\{N_n\}$ denotes the expectation of N_n and $\phi(k)$ is

$$\frac{2([k/2]+1)}{k+1} \binom{k}{[k/2]} 2^{-k} \approx (2\pi k)^{-1/2} .$$

Th 2. With probability one $\underline{\lim}\, N_n / \log n \geq \frac{1}{2}$. We conjecture, but cannot prove, that also $\overline{\lim}\, N_n / (n \log\log n)^{1/2} \leq 1$.

By considering certain subsequences we obtain an exact limit theorem which is still independent of the distribution of the x_i. Let α be positive and a the first integer such that $(1+\alpha)^a \geq 2$; let $1', 2', \cdots$ be any sequence of natural numbers satisfying $(k+1)' \geq (1+\alpha)k'$; and N'_k be the number of changes of sign in $s'_1, \cdots, s'_{k+1}$, where $s'_j = s_{j'}$.

---

* The preparation of this paper was sponsored (in part) by the Office of Naval Research, USN.

** National Bureau of Standards, Los Angeles, and University College of London

*** National Bureau of Standards, Los Angeles, and Cornell University

Th 3. $E\{N'_k\} \geq \frac{1}{8}[k/a]$ and $N'_k / E\{N'_k\} \to 1$

with probability one.

It is likely that $\frac{1}{8}[k/a]$ can be replaced by $\frac{1}{2}\alpha n/(1+\alpha)$. For $k' = 2^k$ it is easy to see that $E\{N'_k\} = k/4$; so with probability one the number of changes of sign in the first k terms of $s_1, s_2, \cdots, s_{2^k}, \cdots$ is asymptotic to $k/4$.

Our proofs are elementary and hardly use more than Lemma 3 of the next section, which gives Theorem 1 immediately. We prove Theorem 3 in §3 and then demonstrate Th 2 by considering particular sequences $1', 2', \cdots$. A sequence $x_1, x_2, \cdots$ for which $N_n / \log n \to 1/2$ is exhibited in §4. And finally we sketch the proof of the following theorem, which was discovered by Paul Levy [1] when the x_i are the Rademacher functions.

Th 4. With probability one

$$\sum_{k=1}^{n} \frac{\operatorname{sgn} s_k}{k} = o(\log n)$$

Our results are stated only for random variables with continuous distributions. Lemma 3, slightly altered to take into account cases of equality, remains true however for discontinuous distributions; the altered version is strong enough to prove the last three theorems as they stand and the first theorem with the extreme members slightly changed. The symmetry of the x_i is of course essential in all our theorems.

2. Let $a_1, \cdots, a_n$ be positive numbers which are free in the sense that no two of the sums $\pm a_1 \pm \cdots \pm a_n$ have the same value. These sums, arranged in decreasing order, we denote by $S_1, S_2, \cdots, S_{2^n}$; q_i is the excess of +'s over -'s occurring in S_i; and $Q_i = q_1 + \cdots + q_i$. It is clear that $Q_i = Q_{2^n - i}$ for $1 \leq i < 2^n$.

Lemma 1 $\quad 0 \leq Q_i - i \leq ([n/2]+1)\binom{n}{[n/2]} - 2^{n-1}$ for $i \leq 2^{n-1}$

The proof of the first inequality, which is evident for $n = 1$, goes by induction. Let $n > 1$ and $i \leq 2^{n-1}$. Define S'_j and Q'_j for $1 \leq j \leq 2^{n-1}$ just as S_j and Q_j above, but using only $a_1, \cdots, a_{n-1}$; and let k and χ be the largest integers such that $S'_k - a_n \geq S_i$ and $S'_\chi + a_n \geq S_i$. It may happen that no such k exists; then $i = \chi$ and

$$Q_i = Q'_\chi + \chi \geq \chi + \chi > i$$

if $\chi \leq 2^{n-2}$,

$$Q_i = Q'_{2^{n-1}-\chi} + \chi \geq 2^{n-1} - \chi + \chi > i$$

if $2^{n-2} < \chi < 2^{n-1}$, and

$$Q_i = Q'_{2^{n-1}} + 2^{n-1} = 2^{n-1} \geq i$$

if $\chi = 2^{n-1}$. The remaining cases are dealt with in the same way.

In order to prove the second inequality we note that for each i the maximum of Q_i is attained if the a_i are given such values that $S_j > S_k$ implies $q_j \geq q_k$, — this happens if the a_j are nearly equal. If n is odd q_i is positive for $i \leq i_0 = 2^{n-1}$ and $Q_i - i$ is maximum for $i = i_0$. We have

$$Q_{i_o} - i_o = \sum_{k=0}^{[n/2]} (n - 2k)\binom{n}{k} - 2^{n-1} = ([n/2] + 1)\binom{n}{[n/2]} - 2^{n-1}$$

A similar computation for n even gives $i_o = 2^{n-1} - \binom{n}{n/2}$ and the same expression for $Q_{i_o} - i_o$. This completes the proof.

If $c_1, \cdots, c_{n+1}$ are real numbers let $m(c_1, \cdots, c_{n+1})$ be the number of indices j for which $|c_j| > |\Sigma_{i \neq j} c_i|$. We now consider n + 1 positive numbers $a_1, \cdots, a_{n+1}$ which are 'free' and define

$$M = M(a_1, \cdots, a_{n+1}) = \Sigma m(\pm a_1, \cdots, \pm a_{n+1}) \quad ,$$

the summation being taken over all combinations of + and - signs.

Lemma 2. $$2^{n+1} \leq M \leq 4([n/2] + 1)\binom{n}{[n/2]} .$$

It is clear that $M = 2^{n+1}$ if $a_{n+1} > a_1 + \cdots + a_n$, and we reduce the other cases to this one by computing the change in M as a_{n+1} is raised to $a_1 + \cdots + a_n + 1$. Let i be the integer between 1 and 2^{n-1} for which $S_{i+1} < a_{n+1} < S_i$ (we use the notation of Lemma 1) and let a'_{n+1} be slightly greater than S_i. Then $a_{n+1} < S_i$ becomes $a'_{n+1} > S_i$ if a_{n+1} is replaced by a'_{n+1} and we see that there is a contribution +4 to M coming from the terms $\pm a_{n+1}$ in the four sums $\pm S_i \pm a_{n+1}$. In like manner, each $+a_j$ occurring in S_i gives -4 to M and each $-a_j$ in S_i gives +4, so that

$$M(a_1, \cdots, a_n, a_{n+1}) - M(a_1, \cdots, a_n, a'_{n+1}) = 4(q_i - 1)$$

Thus raising a_{n+1} to $1 + a_1 + \cdots + a_n$ lowers M by $4(Q_i - i) = 4\Sigma_{j \leq i}(q_j - 1)$ and Lemma 2 follows from Lemma 1.

There is another more direct way of establishing the first inequality of Lemma 2. Considering the n numbers $(a_1 + a_2)$, $a_3, \cdots, a_{n+1}$ we assume that there are at least 2^{n-2} inequalities of the form

$$a_j > U \qquad j > 2 \tag{1}$$

or

$$(a_1 + a_2) > V \tag{2}$$

where the right members are positive, and U is a sum over $(a_1 + a_2), a_3, \cdots, a_{j-1}, a_{j+1}, \cdots, a_{n+1}$ with appropriate signs, and V is a sum over $a_3, \cdots, a_{n+1}$. From (1) we obtain an inequality ($1'$) by dropping the parentheses from $(a_1 + a_2)$ in U; from (2) we obtain ($2'$): $a_1 > a_2 - V$ or $a_1 > V - a_2$ according as a_2 is greater or less than V (we assume without loss of generality that $a_1 > a_2$). We consider also the n numbers $(a_1 - a_2), a_3, \cdots, a_{n+1}$ and the inequalities

$$a_j > U' \qquad j > 2 \tag{3}$$

$$(a_1 - a_2) > V' \tag{4}$$

of which there are at least 2^{n-2}. From (3) we derive ($3'$) by dropping the parentheses from $(a_1 - a_2)$ in U'; from (4) we obtain ($4'$) $a_1 > a_2 + V'$. It is easy to see that no two of the primed inequalities are the same. Hence there must be at least $2 \cdot 2^{n-2} = 2^{n-1}$ inequalities

$$a_i > \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \pm a_j$$

in which the right member is positive. Taking into account the four possibilities of attributing signs to the two members of each inequality we get the first statement of the lemma.

Lemma 3. $$\frac{1}{n+1} \leq \Pr\left\{|x_{n+1}| > |x_1 + \cdots + x_n|\right\} \leq \emptyset(n)$$

Here of course the x_i satisfy the conditions imposed at the beginning of §1 and $\emptyset(n)$ is the function defined there. Since the joint distribution of the x_i is unchanged by permutations of the x_i and multiplication of x_i by -1, we have

$$\Pr\left\{|x_{n+1}| > \left|\sum_{i=1}^{n} x_i\right|\right\} = \frac{1}{n+1} \sum_{i=1}^{n+1} \Pr\left\{|x_i| > \left|\sum_{j\neq i} x_j\right|\right\}$$

$$= \frac{1}{n+1} E\left\{m(x_1, \cdots, x_{n+1})\right\}$$

$$= \frac{1}{n+1} E\left\{\frac{1}{2^{n+1}} \sum_{+,-} m(\pm|x_1|, \cdots, \pm|x_{n+1}|)\right\}$$

$$= \frac{2^{-n-1}}{n+1} E\left\{M(|x_1|, \cdots, |x_{n+1}|)\right\}$$

where m and M are the functions defined above. So Lemma 3 follows at once from Lemma 2.

We cannot prove the inequality

$$\frac{m}{m+n} \leq P_{m,n} \equiv \Pr\left\{\left|\sum_{1}^{n} x_i\right| < \left|\sum_{n+1}^{n+m} x\right|\right\} \leq \emptyset([n/m])$$

for $m \leq n$, which would make our later proofs somewhat easier. But we shall use

$$P_{m,n} \leq 6\emptyset([n/m]) < 3[n/m]^{-1/2} \quad , \tag{5}$$

and establish it in the following manner: Let $a = [n/m]$ and write $z = x_{n+1} + \cdots + x_{n+m}$, $u = x_1 + \cdots + x_{am}$, $v = x_{am+1} + \cdots + x_n$, $w = x_{n+1} + \cdots + x_{am+m}$. It follows from Lemma 3 that the set E on which the four inequalities $|z| < |u \pm v \pm w|$ hold has probability at least $1 - 4\emptyset(a + 1)$ and that the set F on which the two inequalities $|v \pm w| < |u|$ hold has probability at least $1 - 2\emptyset(a)$. So

$$\Pr\{EF\} \geq 1 - 2\emptyset(a) - 4\emptyset(a + 1) > 1 - 6\emptyset(a) \quad .$$

And clearly $|u + v| > |z|$ on EF.

3. It is easy to see that

$$E\{N_n\} = \sum_{k=1}^{n} \Pr\{s_k\, s_{k+1} < 0\} = \frac{1}{2} \sum_{k=1}^{n} \Pr\{|x_{k+1}| > |s_k|\}$$

so that Lemma 3 implies Theorem 1.

Let us turn to Theorem 3. Clearly the probability of s'_k and $s_{2k'} = \Sigma_1^{2k'} x_i$ differing in sign is 1/4. Also $s'_{k+a} - s_{2k'}$ is independent of s'_k and $s_{2k'}$, for $(k + a)' \geq (1 + \alpha)^a k' \geq 2k'$. Since $s'_{k+a} - s_{2k'}$ has an even chance of taking on the same sign as $s_{2k'}$ we must have $\Pr\{s'_k\, s'_{k+a} < 0\} \geq \frac{1}{2} \Pr\{s'_k\, s_{2k'} < 0\} = 1/8$. Now, if $s'_k\, s'_{k+a} < 0$ there must be at least one change of sign in the sequence $s'_k, s'_{k+1}, \cdots, s'_{k+a}$. Hence $p_k + p_{k+1} + \cdots + p_{k+a-1} \geq 1/8$, where $p_k = \Pr\{s'_k\, s'_{k+1} < 0\}$. Consequently

$$E\{N'_k\} = \sum_{1}^{k} p_\chi \geq \frac{1}{8}[k/a] \quad , \tag{6}$$

and the first part of the theorem is proved.

We next show that the variance of N'_k is $O(k)$ by estimating $p_{i,j} = \Pr\{s'_i\, s'_{i+1} < 0 \text{ and } s'_j\, s'_{j+1} < 0\}$ for $i < j$. Let $u = s'_i$, $v = s'_{i+1} - s'_i$, $w = s'_j - s'_{i+1}$, $z = s'_{j+1} - s'_j$ and define the events

A: $uv < 0$

B: $|u| < |v|$

C: $(u + v + w)z < 0$

D: $|u + v + w| < |z|$

D': $|w| < |z|$

E: $|z - w| > |u + v|$,

so that $p_i = \Pr\{AB\}$, $p_j = \Pr\{CD\}$, $p_{i,j} = \Pr\{ABCD\}$. One sees immediately that A, B, C, D' are independent and that ED = ED'. Writing $\tilde{E}$ for the complement of E, we have

$$\begin{aligned} ABCD &= \tilde{E}ABCD + EABCD' \\ &\subset \tilde{E} + ABCD' \end{aligned}$$

and

$$D' \subset \tilde{E} + D \quad .$$

Hence

$$\begin{aligned} \Pr\{ABCD\} &\leq \Pr\{\tilde{E}\} + \Pr\{ABC\}\Pr\{D'\} \\ &\leq \Pr\{\tilde{E}\} + \Pr\{ABC\}(\Pr\{\tilde{E}\} + \Pr\{D\}) \\ &\leq \Pr\{AC\}\Pr\{C\}\Pr\{D\} + 2\Pr\{\tilde{E}\} \\ &= p_i p_j + 2\Pr\{\tilde{E}\} \end{aligned}$$

Now z - w is the sum of $(j+1)' - (i+1)'$ of the x's and u + v is the sum of $(i+1)'$ of the x's; moreover

$$(j+1)' - (i+1)' \geq \left\{(1+\alpha)^{j-i} - 1\right\}(i+1)' \quad .$$

We may thus apply the inequality (5) following Lemma 3 to obtain

$$\Pr\{\tilde{E}\} < 3\left\{(1+\alpha)^{j-i} - 2\right\}^{-1/2}$$

provided $j - i \geq a$. A similar argument gives a lower bound for $p_{i,j}$. We have finally

$$p_{i,j} = p_i p_j + O\left\{(1+\alpha)^{-\frac{1}{2}|i-j|}\right\}$$

for all i,j. This estimate yeilds

$$E\left\{N_k'^2\right\} = \sum_{1\le i,j\le k} p_{i,j}$$

$$= \sum p_i\, p_j + \sum O\left\{(1+\alpha)^{-\frac{1}{2}|i-j|}\right\} \tag{7}$$

$$= E\left\{N_k'\right\}^2 + O(k) \quad .$$

Let us denote $E\left\{N_k'\right\}$ by b_k. It follows from (6), (7) and Tchebycheff's inequality that

$$Pr\left\{\left|N_k'/\, b_k - 1\right| > \epsilon\right| < \frac{c}{\epsilon^2 k}$$

for an appropriate c and all positive ϵ. Hence

$$Pr\left\{\left|\frac{N_{k(j)}'}{b_{k(j)}} - 1\right| > \epsilon\right\}$$

is the j-th term of a convergent series if $k(j) = j^2$, so that $N_{k(j)}'/\, b_{k(j)} \to 1$ with probability one. Since also $b_{k(j)}/\, b_{k(j+1)} \to 1$ and

$$\frac{N_{k(j)}'}{b_{k(j+1)}} \le \frac{N_k'}{b_k} \le \frac{N_{k(j+1)}}{b_{k(j)}}$$

for $k(j) \le k < k(j+1)$, the proof of Theorem 3 is complete.

Theorem 2 is obtained in the following way. Let r be a large integer and let $1'$, $2'$, $\cdots$ be the sequence

$$
\begin{array}{l}
r,\ (r+1), \\
r^2,\ r(r+1),\ (r+1)^2, \\
\quad \cdots \\
r^{\ell},\ r^{\ell-1}(r+1),\ \cdots,\ (r+1)^{\ell}, \\
r^{m},\ r^{m-1}(r+1),\ \cdots,\ (r+1)^{m}, \\
\quad \cdots
\end{array}
$$

with m defined by $r^{m+1} \geq (r+1)^{\ell+1} > r^m$. Let us call j 'favorable' if $(j+1)' = (1+1/r)j'$. Then

a) $(1+1/r)j' \leq (j+1)' \leq (1+r)j'$ for all j

b) There are $k + o(k)$ favorable j less than k (as $k \to \infty$).

We see at once that

$$\log k' = k \log (1+1/r) + o(k) \quad .$$

Furthermore, if j is favorable then $j' = r\{(j+1)' - j'\}$; so, according to Lemma 3,

$$\Pr\{s'_j\ s'_{j+1} < 0\} = \frac{1}{2} \Pr\{|s'_{j+1} - s'_j| > |s'_j|\}$$

$$\geq \frac{1}{2(1+r)} \quad .$$

Hence

$$E\{N'_k\} = \sum_{j=1}^{k} \Pr\{s'_j\ s'_{j+1} < 0\}$$

$$\geq \sum_{j \text{ favorable}} \Pr\{s'_j\ s'_{j+1} < 0\}$$

$$\geq \frac{k}{2(r+1)} + o(k) \quad ,$$

and consequently

$$\lim_{n \to \infty} \frac{2N_n}{\log n} \geq \lim_{k \to \infty} \frac{2N'_k}{\log(k+1)'}$$

$$\geq \lim \frac{2N'_k}{(k+1)\log(1+1/r)}$$

$$\geq \lim \frac{N'_k}{E\{N'_k\}(r+1)\log(1+1/r)}$$

$$= \frac{1}{(r+1)\log 1+1/r} .$$

Letting $r \to \infty$ we have Theorem 2.

4. Our construction of a sequence $x_1, x_2, \cdots$ for which $N_n / \log n \to 1/2$ depends on the following observations. For a given k define the random index $i = i(k)$

$$|x_i| = \max_{1 \le j \le n+1} |x_j|$$

and let A_k be the event $|x_i| > \Sigma\, |x_j|$, where the summation is over $j \ne i$, $1 \le j \le k+1$. Let f_k and g_k be the characteristic functions of the events '$s_k\, s_{k+1} < 0$' and '$i(k) = k+1$ and $x_{k+1}(x_1 + \cdots + x_k) < 0$. It is clear $g_1, g_2, \cdots$ are independent random variables, that $2\Pr\{g_k = 1\} = 1/(k+1)$, and that $f_k = g_k$ on A_k. If moreover $\Sigma\, \Pr\{\tilde{A}_k\} < \infty$ (here $\tilde{A}_k$ is the complement of A_k) then with probability one $f_k = g_k$ for all but a finite number of indices k. In this case we have

$$\begin{aligned} N_n &= \sum_{k=1}^{n} f_k \\ &= \sum_{k=1}^{n} g_k + O(1) \\ &= \sum_{k=1}^{n} \frac{1}{2(k+1)} + o(\log n) \quad , \end{aligned}$$

the last step being the strong law of large numbers applied to $g_1, g_2, \cdots$ Thus we have only to show that the x_j may so be chosen that $\Pr\{\tilde{A}_k\} = O\{k^{-2}\}$, say.

To do this we take $x_j = \pm \exp(\exp 1/u_j)$ where $u_1, u_2, \cdots$ is a sequence of independent random variables each of which is uniformly distributed on the interval $(0, 1)$. For a given k let y and z be

the least and next to least of $u_1, u_2, \cdots, u_{k+1}$; the joint density function of y and z is

$$(k+1)\,k(1-z)^{k-1} \qquad 0 < y < z < 1 \quad .$$

Consequently the event D_k: $1/y > 1/z + 1/k^2$ has probability

$$\int_0^{k^2/(k^2+1)} dy \int_{\frac{y}{1-y/k^2}}^{1} (1-z)^{k-1} dz = 1 + O(k^{-2})$$

and the event E_k: $1/z > 3 \log k$ also has probability $1 + O(k^{-2})$. It is easy to verify that A_k as defined above contains $D_k E_k$; so $\Pr\{\tilde{A}_k\} = O\{k^{-2}\}$ and our example is complete.

5. We prove Theorem 4 in the form

$$T_n = \sum_{\substack{1 \le k \le n \\ s_k > 0}} 1/k = \frac{1}{2} \log n + o(\log n) \quad .$$

First $E\{T_n\} = \sum_1^n 1/k = \frac{1}{2} \log n + o(1)$. Next, the inequality following Lemma 3 yields

$$\Pr\{|s_\ell - s_k| < |s_k|\} = O(\tfrac{k}{\ell})^{1/2} \qquad \ell > k \quad ,$$

so that

$$\Pr\{s_k > 0 \text{ and } s_\ell > 0\} = \frac{1}{4} + O(\tfrac{k}{\ell})^{1/2} \qquad \ell > k \quad .$$

This implies

$$E\{T_n^2\} = \frac{1}{2} \sum_1^n 1/k + 2 \sum_{k<\ell} \{1/4 + O(\tfrac{k}{\ell})^{1/2}\} \frac{1}{k\ell}$$

$$= \frac{1}{4} (\log n)^2 + O(\log n) \quad .$$

Tchebycheff's inequality and the Borel-Cantelli lemma then yield

$$\frac{T_{n_k}}{\log n_k} \to \frac{1}{2} \quad , \tag{8}$$

where $n_k = 2^{k^2}$; and the sequence n_k is dense enough for (8) to imply the theorem.

[1] P. Levy Théorie de l'addition des variables aléatoires Paris 1937.

October 13, 1952

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

Reports and Publications

The results of the Bureau's work take the form of either actual equipment and devices or published papers and reports. Reports are issued to the sponsoring agency of a particular project or program. Published papers appear either in the Bureau's own series of publications or in the journals of professional and scientific societies. The Bureau itself publishes three monthly periodicals, available from the Government Printing Office: The Journal of Research, which presents complete papers reporting technical investigations; the Technical News Bulletin, which presents summary and preliminary reports on work in progress; and Basic Radio Propagation Predictions, which provides data for determining the best frequencies to use for radio communications throughout the world. There are also five series of nonperiodical publications: The Applied Mathematics Series, Circulars, Handbooks, Building Materials and Structures Reports, and Miscellaneous Publications.

Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

NBS